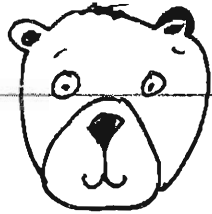


S15 FP2

1. (a) Use algebra to find the set of values of x for which

$$x + 2 > \frac{12}{x + 3}$$

(6)

(b) Hence, or otherwise, find the set of values of x for which

$$x + 2 > \frac{12}{|x + 3|}$$

(1)

a) $(x+2)(x+3)^2 > \dfrac{12(x+3)^{\cancel{2}}}{\cancel{(x+3)}}$

$\Rightarrow (x+2)(x+3)^2 - 12(x+3) > 0$

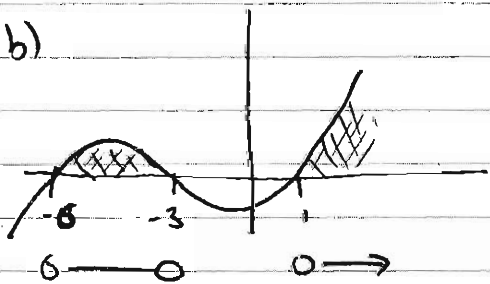
$(x+3)[(x+2)(x+3) - 12] > 0$

$(x+3)(x^2+5x+6-12) > 0$

$(x+3)(x+6)(x-1) > 0$

$-3 \qquad -6 \qquad 1$

b)



>0

$-6 < x < -3$

or

$x > 1$

2.

$$z = -2 + \left(2\sqrt{3}\right)\mathrm{i}$$

(a) Find the modulus and the argument of z.

(3)

Using de Moivre's theorem,

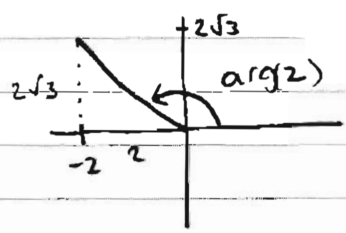
(b) find z^6, simplifying your answer,

(2)

(c) find the values of w such that $w^4 = z^3$, giving your answers in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$.

(4)

a) $|z| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$



$\arg(z) = \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

b) $z = \cos\theta + i\sin\theta \Rightarrow z = 4\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

$z^n = (\cos n\theta + i\sin n\theta \Rightarrow z^6 = 4^6\left[\cos\left(\frac{12\pi}{3}\right) + i\sin\left(\frac{12\pi}{3}\right)\right]$

$\Rightarrow z^6 = 4^6 \times 1 = 4096$

2

c) $z^3 = 4^3\left[\cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right)\right] = 64[1] = 64$

$\therefore w^4 = 64\left[\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)\right]$

$\Rightarrow w = \sqrt[4]{64}\left[\cos\left(\frac{0+2\pi k}{4}\right) + i\sin\left(\frac{0+2\pi k}{4}\right)\right]$

$w = 2\sqrt{2}\left[\cos\left(\tfrac{1}{2}\pi k\right) + i\sin\left(\tfrac{1}{2}\pi k\right)\right]$

$k=0 \quad w_0 = 2\sqrt{2}(\cos 0 + i\sin 0) = 2\sqrt{2}$

$k=1 \quad w_1 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2\sqrt{2}(i) = 2\sqrt{2}i$

$k=2 \quad w_2 = 2\sqrt{2}\left(\cos(\pi) + i\sin(\pi)\right) = 2\sqrt{2}(-1) = -2\sqrt{2}$

$k=3 \quad w_3 = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) = 2\sqrt{2}(-i) = -2\sqrt{2}i$

3. Find, in the form $y = f(x)$, the general solution of the differential equation

$$\tan x \frac{dy}{dx} + y = 3\cos 2x \tan x, \qquad 0 < x < \frac{\pi}{2}$$

(6)

$(\div \tan x)\quad \frac{dy}{dx} + \frac{1}{\tan x} y = 3\cos 2x$

$IF = e^{\int \frac{1}{\tan x} dx} = e^{\int \cot x \, dx} = e^{\ln|\sin x|}$

$= \sin x$

$\Rightarrow \sin x \frac{dy}{dx} + \frac{\sin x}{\tan x} y = 3\cos 2x \sin x$

$\Rightarrow \frac{d}{dx}(y \sin x) = 3\cos 2x \sin x$

$\therefore y\sin x = 3\int \cos 2x \sin x \, dx$ lots of methods to integrate

$y \sin x = 3\int (2\cos^2 x - 1)\sin x \, dx$

$= 3\int 2\sin x \cos^2 x - \sin x \, dx$

$= 6\int \sin x \cos^2 x \, dx \quad - 3\int \sin x \, dx$

$\Rightarrow u = \cos x \qquad\qquad + 3\cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{-du}{\sin x} \qquad \therefore y\sin x = -2\cos^3 x + 3\cos x + c$

$= -6\int u^2 du \qquad \therefore y = \frac{3\cos x - 2\cos^3 x + c}{\sin x}$

$= -\frac{6u^3}{3}$

$= -2\cos^3 x$

Z

4. (a) Show that

$$r^2(r+1)^2 - (r-1)^2 r^2 \equiv 4r^3$$

(3)

Given that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) = (1 + 2 + 3 + \ldots + n)^2$$

(4)

a) $r^2(r+1)^2 - (r-1)^2 r^2$

$= r^2[(r+1)^2 - (r-1)^2] = r^2[\cancel{r^2}+2r+\cancel{1}-\cancel{r^2}+2r-\cancel{1}] = r^2 \times 4r$

$= 4r^3$ #

b) $1^3 + 2^3 + \ldots + n^3 = \sum_1^n r^3$

$\sum_1^n 4r^3 = \sum_1^n r^2(r+1)^2 - (r-1)^2 r^2$

$\therefore \sum_1^n r^3 = \frac{1}{4}\sum_1^n r^2(r+1)^2 - (r-1)^2 r^2$

$= \frac{1}{4}[(1^2 \times \cancel{2^2} - 0) + (\cancel{2^2 \times 3^2} - \cancel{1^2 \times 2^2}) + (\cancel{3^2 \times 4^2} - \cancel{2^2 \times 3^2}) \ldots$

$(\cancel{(n-1)^2 n^2} - \cancel{(n-2)^2(n-1)^2}) + (n^2(n+1)^2 - \cancel{(n-1)^2 n^2})]$

$= \frac{1}{4}n^2(n+1)^2 \quad = [\frac{1}{2}n(n+1)]^2$

$= [\sum_1^n r]^n = (1+2+3+\ldots+n)^2$

$\therefore 1^3 + 2^3 + \ldots + n^3 = (1+2+3+\ldots+n)^2$

#

5. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z}{z + 3i}, \quad z \neq -3i$$

The circle with equation $|z| = 2$ is mapped by T onto the curve C.

(a) (i) Show that C is a circle.

(ii) Find the centre and radius of C.

(8)

The region $|z| \leqslant 2$ in the z-plane is mapped by T onto the region R in the w-plane.

(b) Shade the region R on an Argand diagram.

(2)

a) $w(z+3i) = z \Rightarrow zw + 3wi = z$

$\Rightarrow 3wi = z - zw \Rightarrow z(1-w) = 3iw$

$\therefore z = \dfrac{3iw}{1-w}$

$|z| = 2 \Rightarrow \left|\dfrac{3iw}{1-w}\right| = 2 \Rightarrow |3iw| = 2|w-1|$

$\Rightarrow |i||3w| = 2|w-1|$

$w = u + iv \Rightarrow$ $\Rightarrow |3w| = 2|w-1|$

$\Rightarrow |3u + 3iv| = 2|(u-1) + iv|$

$\Rightarrow |3u + 3iv| = |(2u-2) + 2iv|$

$\Rightarrow 9u^2 + 9v^2 = 4u^2 + 4 - 8u + 4v^2$

$\Rightarrow 5u^2 + 8u + 5v^2 = 4$

$\Rightarrow u^2 + \frac{8}{5}u + v^2 = \frac{4}{5}$

$\Rightarrow \left(u + \frac{4}{5}\right)^2 + v^2 = \frac{4}{5} + \frac{16}{25} = \frac{36}{25} = \left(\frac{6}{5}\right)^2$

Circle, radius $= \frac{6}{5}$ $\left(-\frac{4}{5}, 0\right)$ centre

b)

test $z = 0 + 0i$ $\quad w = \dfrac{0}{0+3i} = 0$ maps to inside.

6.

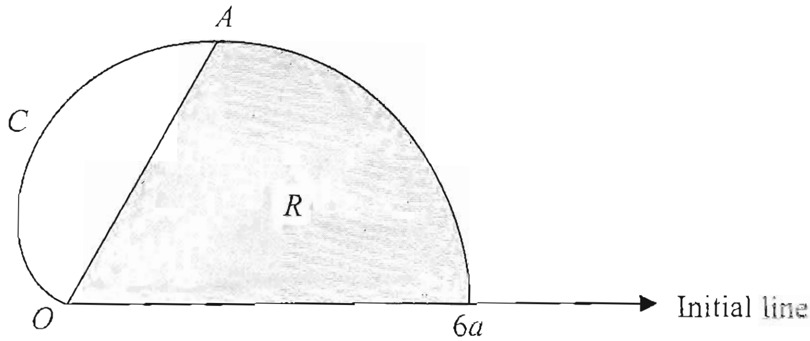


Figure 1

The curve C, shown in Figure 1, has polar equation

$$r = 3a\,(1 + \cos\theta), \qquad 0 \leqslant \theta < \pi$$

The tangent to C at the point A is parallel to the initial line.

(a) Find the polar coordinates of A.

(6)

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line OA.

(b) Use calculus to find the area of the shaded region R, giving your answer in the form $a^2\left(p\pi + q\sqrt{3}\right)$, where p and q are rational numbers.

(5)

$y = r\sin\theta = 3a(1+\cos\theta)\sin\theta$

$y = 3a[(1+\cos\theta)(\sin\theta)]$

$u = 1+\cos\theta \quad v = \sin\theta$

$u' = -\sin\theta \quad v' = \cos\theta$

$\frac{dy}{d\theta} = 3a[-\sin^2\theta + \cos\theta + \cos^2\theta]$

$\frac{d}{dx}uv = vu' + uv'$

$= 3a[\cos^2\theta - 1 + \cos\theta + \cos^2\theta]$

$= 3a[2\cos^2\theta + \cos\theta - 1]$

$= 3a[(2\cos\theta - 1)(\cos\theta + 1)]$

$\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{2} \quad \cos\theta = -1$

$\theta = \frac{\pi}{3} \quad \cancel{\theta = \pi} \quad 0 \le \theta < \pi$

$\therefore r = 3a(1+\cos\frac{\pi}{3})$

$r = 3a \times \frac{3}{2} = \frac{9a}{2}$

$\therefore A\left(\frac{9}{2}a, \frac{\pi}{3}\right)$

Question 6 continued

b) $\text{Area} = \frac{1}{2}\int_{0}^{\alpha} r^2 \, d\theta$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{3}} 9a^2 (1 + (\cos\theta)^2 \, d\theta$$

$$= \frac{9}{2}a^2 \int \cos^2\theta + 2\cos\theta + 1 \, d\theta$$

$$= \frac{9}{2}a^2 \int \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) + 2\cos\theta + 1 \, d\theta$$

$$= \frac{9}{4}a^2 \int \cos 2\theta + 4\cos\theta + 3 \, d\theta$$

$$= \frac{9}{4}a^2 \left[\frac{1}{2}\sin 2\theta + 4\sin\theta + 3\theta \right]_0^{\frac{\pi}{3}}$$

$$= \frac{9}{4}a^2 \left[\left(\frac{1}{2}\sin\frac{2\pi}{3} + 4\sin\frac{\pi}{3} + \pi\right) - (0 + 0 + 0) \right]$$

$$= \frac{9}{4}a^2 \left[\frac{9}{2}\sin\frac{\pi}{3} + \pi \right] = \frac{9}{4}a^2 \left[\frac{9\sqrt{3}}{4} + \pi \right]$$

$$= a^2 \left[\frac{81\sqrt{3}}{16} + \frac{9\pi}{4} \right]$$

7.

$$y = \tan^2 x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(a) Show that $\dfrac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x$

(4)

(b) Hence show that $\dfrac{d^3y}{dx^3} = 8\sec^2 x \tan x \,(A \sec^2 x + B)$, where A and B are constants to be found.

(3)

(c) Find the Taylor series expansion of $\tan^2 x$, in ascending powers of $\left(x - \frac{\pi}{3}\right)$, up to and including the term in $\left(x - \frac{\pi}{3}\right)^3$

(4)

$y = \tan^2 x \Rightarrow \dfrac{dy}{dx} = 2(\tan x)^1 \times \sec^2 x$

$\Rightarrow \dfrac{dy}{dx} = 2\tan x \sec^2 x$

$u = \tan x \qquad v = (\sec x)^2$

$u' = \sec^2 x \qquad v' = 2\sec x \times \sec x \tan x$

$= 2\sec^2 x \tan x$

$\dfrac{d^2y}{dx^2} = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$

$= 6\sec^4 x - 4\sec^2 x$ #

b) $\dfrac{d^3y}{dx^2} = 24\sec^3 x \times \sec x \tan x - 8\sec x \times \sec x \tan x$

$= 24\sec^4 x \tan x - 8\sec^2 x \tan x$

$= 8\sec^2 x \tan x\,(3\sec^2 x - 1)$ $\quad A = 3$

$B = -1$

2

c) $x = \frac{\pi}{3}$ $\quad \tan x = \tan\frac{\pi}{3} = \sqrt{3}$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$y = \tan^2 x = \left(\sqrt{3}\right)^2 = 3$$

$$\frac{dy}{dx} = 2\tan x \sec^2 x = 2(\sqrt{3})(2)^2 = 8\sqrt{3}$$

$$\frac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x = 6(2)^4 - 4(2)^2 = 80$$

$$\frac{d^3y}{dx^3} = 8\sec^2 x \tan x\left(3\sec^2 x - 1\right) = 8(2)^2(\sqrt{3})\left(3(2)^2 - 1\right) = 352\sqrt{3}$$

$$\therefore \tan^2 x \simeq 3 + 8\sqrt{3}\left(x - \frac{\pi}{3}\right) + 40\left(x - \frac{\pi}{3}\right)^2 + \frac{352}{6}\sqrt{3}\left(x - \frac{\pi}{3}\right)^3$$

8. (a) Show that the transformation $x = e^u$ transforms the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = 2 \ln x, \qquad x > 0 \qquad \text{(I)}$$

into the differential equation

$$\frac{d^2 y}{du^2} - 8 \frac{dy}{du} + 16y = 2u \qquad \text{(II)}$$

(6)

(b) Find the general solution of the differential equation (II), expressing y as a function of u.

(7)

(c) Hence obtain the general solution of the differential equation (I).

(1)

a) $x = e^u \Rightarrow \frac{dx}{du} = e^u = x$

$\ln x = u \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{x}\frac{dy}{du}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{du}\right]$

$u = \frac{1}{x}$ $\quad v = \frac{dy}{du}$

$u' = \frac{-1}{x^2}$ $\quad v' = \frac{d}{dx}\left(\frac{dy}{du}\right)$

$= \frac{du}{dx} \times \frac{d}{du}\left(\frac{dy}{du}\right)$

$= \frac{1}{x} \times \frac{d^2y}{du^2}$

$= \frac{1}{x^2}\frac{d^2y}{du^2} - \frac{1}{x^2}\frac{dy}{du}.$

$x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 2\ln x$

$x^2\left(\frac{1}{x^2}\frac{d^2y}{du^2} - \frac{1}{x^2}\frac{dy}{du}\right) - 7x\left(\frac{1}{x}\frac{dy}{du}\right) + 16y = 2u$

$\Rightarrow \frac{d^2y}{du^2} - \frac{dy}{du} - 7\frac{dy}{du} + 16y = 2u \quad \therefore \frac{d^2y}{du^2} - 8\frac{dy}{du} + 16y = 2u$ #

Question 8 continued

b) PI $y = au + b$ $\quad y'' - 8y' + 16y \equiv 2u$

$y' = a$ $\quad 0 - 8a + 16au + 16b \equiv 2u$

$y'' = 0$

$16au \equiv 2u \quad \therefore a = \frac{1}{8}$

$16b - 8a \equiv 0$

$y_{PI} = \frac{1}{8}u + \frac{1}{16} \quad 16b = 1 \quad \therefore b = \frac{1}{16}$

CF $y = Ae^{mu}$ $\quad y'' - 8y' + 16y = 0$

$y' = Ame^{mu}$ $\quad Am^2e^{mu} - 8Ame^{mu} + 16Ae^{mu} = 0$

$y'' = Am^2e^{mu}$

$\underbrace{Ae^{mu}}_{\neq 0}(m^2 - 8m + 16) = 0$

$(m-4)^2 \quad \therefore m = 4$

$\therefore y_{cf} = (A + Bu)e^{4u}$

$\therefore y_{GS} = (A + Bu)e^{4u} + \frac{1}{8}u + \frac{1}{16}$

c) $u = \ln x$ $\quad y = (A + B\ln x)e^{4\ln x} + \frac{1}{8}\ln x + \frac{1}{16}$

$y = (A + B\ln x)(e^{\ln x})^4 + \frac{1}{8}\ln x + \frac{1}{16}$

$y = (A + B\ln x)x^4 + \frac{1}{8}\ln x + \frac{1}{16}$

Z